Math 261B Thurs. 9/3

G group $\rightsquigarrow$ $A = \mathcal{O}(G)$ Hopf algebra

$G \curvearrowright V = K^n \quad \longleftrightarrow \quad W = V^*$ is a (right) comodule for A:

$$W \xrightarrow{\rho} W \otimes A$$

$R = \mathcal{O}(V) = k[W]$

$\quad = k[x_1, \dots, x_n]$

$v_1, \dots, v_n$ basis of V

$v = x_1 v_1 + \dots + x_n v_n \rightsquigarrow \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$G \times V \to V \quad \leftrightsquigarrow \quad R \to R \otimes A$

$$\begin{array}{cc} G & V \\ \Downarrow & \Downarrow \\ g & v \\ & \parallel \\ M_{ij}(g) & \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \end{array}$$

$x_i \mapsto (M_{ij}\vec{x})_i$

$\quad = \sum_j M_{ij} x_j$

$\quad = \sum_j \underbrace{x_j}_{\in R} \otimes \underbrace{M_{ij}}_{\in A} \qquad M_{ij}(g)$

$x_1, \dots, x_n$ basis of $V^* = W$

$$\begin{array}{ccc} G \times G \times V & \longrightarrow & G \times V \\ \downarrow & & \downarrow \\ G \times V & \longrightarrow & V \end{array}$$

$W \mapsto W \otimes A$ is the coaction $\leftarrow$

Δ in A makes A^* an algebra : $A^* \otimes A^* \hookrightarrow (A \otimes A)^* \xrightarrow{\Delta^*} A^*$

(with unit $\varepsilon : A \to K$)

$G \hookrightarrow A^*$

$g \mapsto ev_g$

$W \xrightarrow{\rho} W \otimes A \xrightarrow{id_W \otimes ev_g} W \otimes K = W$

$g \in G \subset A^*$

What about infinite dimensional comodules?

$W \xrightarrow{\rho} W \otimes A \qquad x \in W \qquad \rho(x) = \sum_i x_i \otimes a_i$ (x_i in a chosen basis of W)

Coaction $\Rightarrow$

$$\sum_i x_i \otimes a_{i,1} \otimes a_{i,2} = \sum_{i,j} x_{i,j} \otimes m_{ij} \otimes a_i$$

If we take the $x_{i,j}$ from the chosen basis, they are among the x_i.

$\rho(x_i) = \sum_j x_{ij} \otimes m_{ij}$

$W' =$ span of x_i's in $\rho(x)$ is a finite dimensional subcomodule $\rho|_{W'} \subset W' \otimes A$.

$\Rightarrow$ Every comodule is locally finite

G affine $\Rightarrow$ $A = \mathcal{O}(G)$ is a Hopf alg. $\quad \Delta : A \to A \otimes A$

"w" $\quad$ "w"

makes A a faithful right comodule

locally finite

$G \curvearrowright A \qquad (\underset{G}{g} \cdot \underset{A}{f})(\underset{G}{h}) = f(hg)$

$$G \xrightarrow{\cdot g} G$$
$$A \leftarrow A$$

Ex. $\quad \mathbb{G}_a = (k, +) \qquad A = k[z]$

$x \in k = \mathbb{G}_m \qquad x \cdot f(z) = f(z + x)$

$k \cdot \{1, z, \dots, z^m\}$ is invariant

Also unipotent: $\quad x \cdot z^k = (z+x)^k = z^k + \binom{k}{1} z^{k-1} x + \cdots$

x acts by matrix $\begin{pmatrix} 1 & \vdots & & \\ 0 & \ddots & & \\ 0 & 1 & & \\ 0 & 0 & \ddots & \\ 0 & 0 & & \\ 0 & 0 & & \end{pmatrix}$ $\quad = z^k +$ lower terms.

upper uni-triangular.

$k \cdot 1$ is trivial, $\quad k \cdot \{1, z\} : \quad x \mapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$

$1, z \qquad z + x$

Ex. $\mathbb{G}_m = (K^\times, \cdot) = GL_1$ $\quad A = K[t^{\pm 1}]$ $\quad a \in K^\times = \mathbb{G}_m$ $\quad a \cdot f(t) = f(ta)$

$f \in k[t^{\pm 1}]$

$t^m \underset{ta}{\mapsto} a^m t^m$

$K \cdot \{t^m\}$ is a 1-dim'l rep with

$a \mapsto (a^m)$

$m = \pm 1$ are faithful 1-dim'l reps.

• $x_1, \dots, x_n$

Geometric lemmas

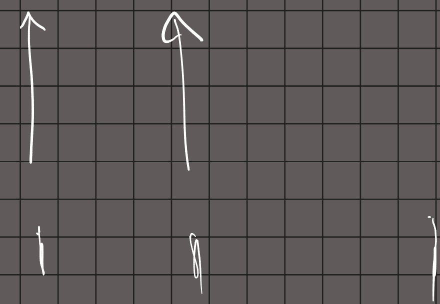
o) G is smooth

$\Rightarrow$ irr components are the connected components

$e \mapsto g$

$G \xrightarrow{g\cdot} G$

Ex. $xy = 0$ in K^2



$G \overset{\text{finite}}{\supset} G_0$

1) $H \subset G$ subgroup $\Rightarrow \overline{H}$ is a subgroup.

$h \in H$ $\quad h \cdot (-) : G \to G$ is cont's

H goes into H $\Rightarrow \overline{H}$ into $\overline{H}$

$H\overline{H} \subset \overline{H}$ same argument with $h' \in \overline{H}$ $\Rightarrow \overline{H} \cdot \overline{H} \subset \overline{H}$

$i(H) \subset H \Rightarrow i(\overline{H}) \subset \overline{H}$.

2) Every constructible subgroup of G is closed

$\overset{\shortparallel}{\text{finite}}$ union of locally closed subsets

$\overset{\shortparallel}{\text{(open)}} \cap \text{(closed)}$.

H constructible $\Rightarrow H$ contains some non-empty open subset of $\overline{H}$.

$H, \overline{H}$ subgroups $\Rightarrow H$ is open in $\overline{H}$ $\overset{?}{\Rightarrow}$ H is closed in $\overline{H}$

$H = \overline{H}$.

3) Image of any morphism is constructible.

$G \xrightarrow[\varphi]{} G'$ hom of alg. grps. ker φ is a closed (normal) subgroup of G

im φ is a closed subgp of G'

4) If G linear, $A = \mathcal{O}(G)$, H closed normal subgroup

$\Rightarrow$ G/H is linear, $\mathcal{O}(G/H) = A^H = \{f \in A : hf = f \ \forall h \in H\}$

$$(h \cdot f)(g) = f(gh)$$

$A^H \supseteq$ f constant on cosets of H

What if H = non-normal closed subgp $H \subseteq G$ $\Rightarrow$ G/H is a smooth alg. variety

not necessarily affine!

Ex. $\underset{\text{upper uni}\Delta\text{'ar}}{U} \subseteq \overset{\text{upper }\Delta\text{'ar}}{B} \subseteq GL_n$ $\qquad B/U$

$U = \ker(B \to T)$, T = diagonal $\cong (K^\times)^n = (\mathbb{G}_m)^n$

$$\begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \Big/ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$n = 2$ $\quad \begin{pmatrix} t_1 & x \\ 0 & t_2 \end{pmatrix} \quad \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \in U \quad \rightsquigarrow f(t_1, t_2, x)$

$A = k[t_1^{\pm 1}, t_2^{\pm 1}, x]$
$\quad = \mathcal{O}(B)$

$$\begin{pmatrix} t_1 & x \\ 0 & t_2 \end{pmatrix}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t_1 & at_1 + x \\ 0 & t_2 \end{pmatrix}$$

$$a \cdot f = f(t_1, t_2, x + at_1)$$

$A^U = k[t_1^{\pm 1}, t_2^{\pm 1}]$?

$f(t_1, t_2, x + a t_1) = f(t_1, t_2, x) \quad \forall a$

$\Updownarrow$

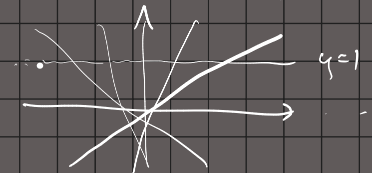
f independent of x

$B/U = T$ with $\mathcal{O}(T)$

Ex. GL_n/B

$GL_2 \Big/ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} = \{1\text{-dim'l subspaces of } K^2\}$

$= \mathbb{P}^1(K)$



$GL_n \curvearrowright K^n$

$\curvearrowright \{\text{linear subspaces } V \subseteq K^n \text{ s.t. } \dim V = d\}$

$\curvearrowright \{\text{flags } 0 \subset V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset K^n : \dim V_i = i\}$

transitively

Standard flag: $V_i = \langle e_1, \dots, e_i \rangle$

$$\begin{pmatrix} * & * & & \\ 0 & * & & \\ \vdots & 0 & \ddots & \\ \vdots & \vdots & & \ddots \\ 0 & \vdots & & \end{pmatrix}$$

V_1

Stab of ( ) is B

$G/B =$ set of flags $=$ projective alg variety ...

$\mathbb{P}^1 \setminus$ any point is $\cong A^1 = K$ $\quad \mathcal{O}(A^1) = k[x]$

Only global alg. functions $f \in \mathcal{O}(\mathbb{P}^1)$ are constants!

$\mathbb{P}^1$ is projective, not affine.